INCORRECT MATHEMATICS IN LAW RESEARCH: A COMMENT[#]

CHRISTIAN SEIDL[*]
University of Kiel, Germany, Department of Economics

Ruben Martini tried to compare corporation tax régimes of different countries by means of elasticity, a concept imported from economics. Alas, his analysis teems with errors, such as subtraction of qualitative variables instead of numerical variables, lack of quantitative relationships between dependent and independent variables, neglect of approximating the derivative of the relationship between dependent and independent variables by infinitely small distances of variables by choosing arbitrarily different values of the variables, lack of comparability of the involved variables, etc. Rigorous methods may turn out as very prolific for law research, provided that their rules are respected. Martini's paper constitutes serious violations of proper use of mathematics.

Ruben Martini 2013 published a paper on a mathematical model of elasticity in application to international legal comparisons of corporation tax régimes. But instead of providing a promising methodological innovation in law research, his paper teems with mathematical and logical errors.

Martini 2013, 517-518, started to define elasticity by and large not incorrectly. He asserted that «the mathematical concept of elasticity describes the changes of variables that are functionally connected». He used a function $y = f(x)$ and described its elasticity ε_{xy} by

$$(1) \qquad \varepsilon_{xy} = \frac{f'(x)x}{f(x)} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \frac{x}{f(x)} \approx \frac{\frac{\Delta y}{y}}{\frac{\Delta x}{x}}.$$

He called y the dependent variable and x the parameter, which is contrary to mathematical usage, because parameters are required for describing the shape of function $f(x)$, not the independent variable x. Even calling y the dependent variable and x the independent variable implies the notion of causality which may reflect the imagination of the beholder, but does not result from the very elasticity formula. If $f(x)$ is invertible, then the inverse elasticity can be formulated for the inverse function $x = f^{-1}(y)$. Another beholder may well consider x as the dependent and y as the independent variable and use the inverse elasticity.[1]

While Martini 2013, 517-518, acknowledged that Δy and Δx should be infinitely small for the definition of point elasticity and very small for its approximation, he alienated the elasticity concept in his application to comparative corporation tax law. He (2013, 524) defined four variables, viz. $y_{ij}, y_{k\ell}, x_{ij}, x_{k\ell}$. y_{ij} denotes the corporation tax system in jurisdiction (i.e. country) i at time j, $y_{k\ell}$ denotes the corporation tax system in jurisdiction (i.e. country) k at time ℓ, x_{ij} denotes the private law system prevailing in jurisdiction (i.e. country) i at time j, and $x_{k\ell}$ denotes the private law system prevailing in jurisdiction (i.e. country) k at time ℓ. This notation is somewhat puzzling: does it, e.g., make much sense to compare the legal system in USA for the year 2011 with the German legal system in the year 1929?

Alienating the elasticity formula, Martini 2013, 524, defined a rather peculiar type of elasticity:

[#] I am indebted to two anonymous referees for constructive and helpful critique.

[*] Address for correspondence: Christian Seidl (Professor emeritus, Department of Economics, University of Kiel), Olhausenstraße 40, D-24098 Kiel (Germany); e-mail: seidl@economics.uni-kiel.de.

[1] One of the referees suggested the following explanation: «There is no causation between x and y. Causation is a statistical concept. It does not fit real analysis.»

$$(2) \qquad \varepsilon_{x_{ij}x_{k\ell}y_{ij}y_{k\ell}} = \frac{\frac{y_{ij}-y_{k\ell}}{y_{k\ell}}}{\frac{x_{ij}-x_{k\ell}}{x_{k\ell}}}.$$

The prerequisites of calculating such an elasticity are:

1. A functional relationship between the x's and the y's should be defined.
2. All variables should be measurable in terms of real numbers.
3. The variables in the numerator should have the same unit of measurement and the variables in the denominator should have the same unit of measurement (not necessarily the same). Otherwise, subtraction and division cannot be carried out. Since both the (composed) numerator and the (composed) denominator of (2) are unitless, their division is of course feasible.
4. Elasticities as point elasticities in $\tilde{x}$, $\varepsilon(\tilde{x})$, are just the ratio of the slope of the function $f(x)$ at point $\tilde{x}$, $f'(\tilde{x})$, and its average $f(\tilde{x})/\tilde{x}$ at point $\tilde{x}$. It indicates whether a slight absolute change of $f(x)$ in the neighborhood of $f(\tilde{x})$ is higher or smaller than an absolute proportional change of $f(x)$. The approximation of a point elasticity by a so-called bow elasticity $\frac{\Delta y}{y}/\frac{\Delta x}{x}$ requires that the two components of Δy and Δx are each close together. Otherwise the slope $f'(x)$ cannot be approximated.
5. There are two possibilities for the numerator and the denominator of (2), viz.

$$\frac{y_{ij}-y_{k\ell}}{y_{k\ell}} \text{ or } \frac{y_{ij}-y_{k\ell}}{y_{ij}} \text{ and } \frac{x_{ij}-x_{k\ell}}{x_{k\ell}} \text{ or } \frac{x_{ij}-x_{k\ell}}{x_{ij}}.$$

   With respect to the numerator, Martini 2013, 524, argued: «There is, however, a need for a point of reference. Without such a point, the numerical value of the change of the dependent variable cannot be precisely determined.» Since both y_{ij} and $y_{k\ell}$ are equipollent candidates for points of reference, the choice for one of these candidates is arbitrary; choosing the other value would normally give different results.

All five prerequisites of calculating an elasticity are absent in Martini's analysis. Notwithstanding this did not prevent Martini from making calculations.

With respect to the denominator of (2) Martini 2013, 526, distinguished but two cases. Either the private law systems of the jurisdictions to be compared are «congruent» (i.e. virtually identical), then he set $\frac{x_{ij}-x_{k\ell}}{x_{k\ell}} = 0$, or they are different, then he set $\frac{x_{ij}-x_{k\ell}}{x_{k\ell}} = -1$.

Although Martini failed to reveal any method as to how the variables can be measured and thus compared, he (525) distinguished for the numerator four conditions:
«endogenous adaptions»

$$(3) \qquad \frac{y_{ij}-y_{k\ell}}{y_{k\ell}} > 0,$$

«congruence»

$$(4) \qquad \frac{y_{ij}-y_{k\ell}}{y_{k\ell}} = 0,$$

«partial replacement»

$$(5) \qquad -1 < \frac{y_{ij} - y_{k\ell}}{y_{k\ell}} < 0,$$

«total replacement»

$$(6) \qquad \frac{y_{ij} - y_{k\ell}}{y_{k\ell}} = -1.$$

Then Martini 2013, 530, summarized his results for equation (2). For $\frac{x_{ij}-x_{k\ell}}{x_{k\ell}} = 0$, the elasticity is not defined for (3), (5), and (6). Martini noted that it approaches infinity and is «perfectly inelastic». To $\frac{x_{ij}-x_{k\ell}}{x_{k\ell}} = 0$ and (4) he ascribes «qualitative indication for an elastic relation». No mathematician would understand that $\frac{0}{0}$ is a qualitative indication for an elastic relation.

For $\frac{x_{ij}-x_{k\ell}}{x_{k\ell}} = -1$ the elasticity is «perfectly inelastic" for (3) and (4)[2], «relatively inelastic» for (5), and «unitary elastic» for (6). In view of obvious noncomparability of y_{ij} and $y_{k\ell}$, it is puzzling how he could distinguish these cases. He (530) remarked on this case: «The relation between the dependent variable and the parameter represents the grade of the connection between them.» It is mysterious to refer to a «grade of connection» between the dependent and the independent variable (the latter called «parameter» by Martini), since the independent variable has, according to Martini, the constant value -1 throughout, as $x_{ij} \neq x_{k\ell}$. Rather the variables (3) to (6) are independent of the constant -1; neither does a dependence on $\pm\infty$ exist.

For another case Martini 2013, 533, derived somewhat different results. For $x_{ij} = x_{k\ell}$, the numerator in (2) is again divided by zero and comes up to $\pm\infty$. Martini also considers a case $x_{ij} \neq y_{k\ell}$ and argued that this holds for a «change from real seat theory to incorporation theory or vice versa». This move is not substantiated by a working example and, therefore, remains rather obscure. Normally an unbiased observer would assume that the corporation tax régime and the private law system in different jurisdictions and in different times are of course different. But, apart from the lack of comparability, the results do not seem to be encouraging: the elasticities in the latter case are either one or zero.

So far, I followed Martini in considering (2) to be a «rather peculiar type of elasticity», but, nevertheless, elasticity. Only in prerequisite No. 4 did I express some proviso. Coming back to my remark that Martini alienated the elasticity concept, the question whether (2) can be called elasticity should be scrutinized. Put in a nutshell, elasticity is nothing but the ratio of the derivative or slope of function $y = f(x)$, $f'(x)$, and its average $f(x)/x$. This ratio of two functions, viz. $f'(x)/\frac{f(x)}{x}$, evaluated, say at $\tilde{x}$, then just denotes the elasticity at point $\tilde{x}$. In Martini's «elasticity» (2), no derivative can be ascertained.

---

[2] (4) and $\frac{x_{ij}-x_{k\ell}}{x_{k\ell}} = -1$ comes up to $\frac{0}{0}$ again.

Elasticity can be approximated by differences $\frac{\Delta y}{y} / \frac{\Delta x}{x}$, but these differences have to be small for providing a good approximation. Note that this approximation is very convenient for aggregate values, in particular for macroeconomics, since it is nothing else than the ratio of two growth rates. Indeed, this is the reason for major application of elasticities in empirical economic investigations.

However, (2) cannot be addressed as elasticity. Bick and Sydsæter 1991, 17-20, presented a comprehensive catalogue of 28 types of elasticities, but something, which is even remotely akin to (2), is not contained in their catalogue. Rather (2) is the ratio of two relative differences, provided the above prerequisites are satisfied. Note that the relative differences can be normalized in two ways. Focusing on one way only needs a convincing justification, which Martini failed to provide.

Martini 2013, 519, tried to justify his approach by remarks such as:

> The process of translating law into numbers is merely transitional. It does not represent a self-contained analysis. The idea of elasticity is only to be used as a numerical model to inspire a jurisprudential and therefore qualitative approach to description and systematization of results of comparative legal study. Such a usage of mathematics does not mean a quantitative analysis of law. Unlike empirical legal studies, the elasticity method does not directly deal with the relationship between law and reality. Rather, it is used as a translational model. In contrast to quantitative methods, the numerical approach is based on data that is not derived from empirical research, but from the transfer of language into numbers.[3]

Hence, Martini confessed that the comparison of two tax mechanisms is nothing else than the beholder's perception concerning the degree of similarity between the two mechanisms.[4] Why then employ mathematics to express a simple qualitative statement? The use of mathematics should dupe kind of rigorous precision, which is completely absent if symbols without any real content and violating the rules of mathematical reasoning are used. It imparts the impression that readers should just be buffaloed by using a rigorous method to obscure that, for questions of this kind, common sense would be the proper method. Science should avoid any semblance to a dodge.

The critique expressed in this comment should not convey the impression that mathematics cannot be put to good use in law research. For instance, it is revealing to compare the ratio of the growth rates of corporation tax revenue and GDP for two countries. This shows whether the elasticity of corporation tax revenue with respect to GDP is more sensitive in one country than in another country. The ratio of these elasticities indicates the relative sensitivity of corporation tax revenue comparing the two countries.[5] Another use of elasticities may consist of the ratio of the growth rate of the capitalized value of all joint stock companies and the growth rate of GDP or domestic wealth and compare that for different countries. This would

---

[3] MARTINI 2013, 523, also stated: «Thus, unlike mathematical elasticity, comparative elasticity is not based on ratio scales, but on interval scales. … In comparative law, elasticity fulfils a mere systemizing and documenting function … For elasticity to fulfill this function, it does not need to operate on a ratio scale, but can operate on a nominal scale.» This shows that Martini is unaware of measurement scales. A good outline of measurement scales was, e.g., provided by STEVENS 1975, 47-51.

[4] Martini brought his 2013 model also in his 2016 dissertation. The latter is not referred to in this comment, since, if at all, it was but insignificantly changed and is in German. Readers of HEI can better check my comment consulting the earlier English version.

[5] For income tax, SEIDL ET AL. 2013 analyzed the comparison of tax progression in OECD countries.

show the difference in relative increasing importance of the corporate sector in the countries to be compared. One can also measure elasticities as the ratio of the growth rate of innovations of new patents and the growth rate of the capitalized value of joint stock companies. This can be compared between two countries to look for tendencies of relatively better innovations in the performance of the respective countries. There are indeed manifold applications of proper elasticities which can easily be measured. Mathematical methods may also be usefully applied to analyze criminal law.[6] Prudentially used, mathematics may be highly prolific for law research.

References

BICK P. and SYDSÆTER K. 1991, *Economists' Mathematical Manual,* Berlin etc., Springer Verlag.

MARTINI R. 2013, «Numerical Methodology in Comparative Tax Law: The Mathematical Model of Elasticity as a Thinking Model for Legal Comparisons», *Cambridge Journal of International and Comparative Law*, 2, 2, 506-535. [This journal has meanwhile changed its title to *Cambridge International Law Journal*. It has also changed its focus and, according to its editor, does no longer deal with international comparisons of taxation. It seems that its refereeing, at least in 2013, was somewhat defective.]

MARTINI R. 2016, *Der persönliche Körperschaftsteuertatbestand: Eine rechtsvergleichend-historische Analyse der Bestimmung von eigenständig steuerpflichtigen Personenvereinigungen*, Tübingen, Mohr Siebeck.

SEIDL C. 2016, «Bemerkungen zum Gleichheits- und Bestimmtheitsgebot von Geldstrafen: Eine kritische Analyse», *Rechtswissenschaft – Zeitschrift für rechtswissenschaftliche Forschung*, 7, 2, 211-269.

SEIDL C., POGORELSKIY K., and TRAUB S. 2013, *Tax Progression in OECD Countries: An Integrative Analysis of Tax Schedules and Income Distributions*, Heidelberg, Springer-Verlag.

STEVENS S. S. 1975, *Psychophysics: Introduction to Its Perceptual, Neural, and Social Prospects*, New York, Wiley.

[6] Cf. SEIDL 2016 for German criminal law.