Homework Week Seven

Part 1 Mathematics

Def 1.1 Sequence
Suppose M is a set. A sequence in M is a mapping $f : \mathbb{N} \to M$.
For the images $f(1), f(2), \ldots$ we will write $x_1, x_2, \ldots$.
We will denote the sequence by the symbol $(x_n)_{n\in\mathbb{N}}$.
Def 1.1 Convergence of a sequence
(1) A sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ is called convergent with limit $x \in \mathbb{R}$ if

$$\forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \forall n \geq N_\epsilon : \quad |x_n - x| \leq \epsilon.$$

We'll denote this by $x = \lim_{n\to\infty} x_n$
(2) A sequence $(x_n)_{n\in\mathbb{N}} in \mathbb{R}$ is called convergent, if there exists a limit x to which it converges, i.e

$$\exists x \in \mathbb{R} \forall \epsilon > 0 \exists N_\epsilon \in \mathbb{N} \forall n \geq N_\epsilon : \quad |x_n - x| \leq \epsilon.$$

Fundamental Lemma 1.3 Archimedean property of $\mathbb{R}$
Suppose $\epsilon > 0$. There exists a $N_\epsilon \in \mathbb{N}$ such that $\frac{1}{N_\epsilon} \leq \epsilon$.
We will not prove this!
Examples 1.4
(1) *The sequence $(\frac{1}{n})_{n\in\mathbb{N}}$ converges to 0.*
Proof: Suppose $\epsilon > 0$. The fundamental lemma gives us the existence of a $N_\epsilon \in \mathbb{N}$ such that $\frac{1}{N_\epsilon} \leq \epsilon$. For $n \geq N_\epsilon$ we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N_\epsilon} \leq \epsilon$$

Thus the sequence converges to 0.
(2) *The sequence $(x_n)_{n\in\mathbb{N}}$ with $x_n = 1000$ if $n = 1000$ and $x_n = \frac{1}{n}$ else, converges to 0.*
Proof: Suppose $\epsilon > 0$. The fundamental lemma gives us the existence of a $N_\epsilon \in \mathbb{N}$ such that $\frac{1}{N_\epsilon} \leq \epsilon$. Let $\tilde{N}_\epsilon := \max\{1000, N_\epsilon\}$. For $n \geq \tilde{N}_\epsilon$ we have

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N_\epsilon} \leq \epsilon$$

Thus the sequence converges to 0.
(3) *The sequence $\left(\frac{1}{n^2}\right)_{n\in\mathbb{N}}$ converges to 0.*

Proof: Suppose $\epsilon > 0$. The fundamental lemma gives us the existence of a $N_\epsilon \in \mathbb{N}$ such that $\frac{1}{N_\epsilon} \leq \epsilon$. For $n \geq N_\epsilon$ we have

$$\left| \frac{1}{n^2} - 0 \right| = \frac{1}{n^2} = \frac{1}{n} \cdot \frac{1}{n} \leq \frac{1}{N_\epsilon} \cdot \frac{1}{n} \leq \frac{1}{N_\epsilon} \leq \epsilon$$

Thus the sequence converges to 0.

Exercises 1.5

Prove the following:

(1) The sequence $\left(1 - \frac{1}{n}\right)_{n\in\mathbb{N}}$ converges to 1.

(2) The sequence $\left(\frac{2}{5n^2+6}\right)_{n\in\mathbb{N}}$ converges to 0.

(3) The sequence $\left(\frac{2^n n!}{n^{n+1}}\right)_{n\in\mathbb{N}}$ converges to 0.

Hint: For $n \geq 6$ we have $2^n \leq \frac{n^n}{n!}$. You don't have to prove this. But you can easily do it by induction.

Please prove the following proposition:

Proposition 1.6

A convergent sequence in $\mathbb{R}$ has a unique limit.

Hint: Write out the definition for two possible candidates and apply the triangle inequality.

Part 2 Economics

2.1 Properties of the cost function

You are a price taker on the input market and you want to minimize expected cost for a given output y. You may choose to locate in one of two countries, a or b. In country a the price vector is $(1, 1, 1)$, in country b it is $(1.5, 1, 1)$ with probability $1/2$ and $(0.5, 1, 1)$ with probability $1/2$. Would you chose a or b?
Hints: You don't need to assume anything concerning risk preferences here, simply use the properties of the cost function.

2.2 True or False? Give a proof if one of the following propositions is true, otherwise give a counterexample. The definitions are stated on page 96 of Varian.

1. If preference satisfy *local nonsatiation*, then they satisfy *weak monotonicity.*

2. If preference satisfy *weak monotonicity*, then they satisfy *local nonsatiation.*

3. If preference satisfy *strong monotonicity*, then they satisfy *local nonsatiation.*

4. For differentiable utility $u : \mathbb{R}^n_+ \to \mathbb{R}$ *strong monotonicity* is equivalent to having $\partial u/\partial x_i \geq 0 (> 0)$ for at least one $i \in \{1, \ldots, n\}$.

5. For differentiable utility $u : \mathbb{R}^n_+ \to \mathbb{R}$ *strong monotonicity* is equivalent to having $\partial u/\partial x_i \geq 0 (> 0)$ for all $i \in \{1, \ldots, n\}$.

2.3 Expenditure and indirect utility

Show that, under *strong monotonicity*, one has

$$y = e(p, v(p, y)).$$

Show that *weak monotonicity* is not sufficient for this equality to hold true identically.

Please hand this in on Monday (12.12.2015).